LINEAR ALGEBRA : Basics for Regression Analysis

Theorem 4

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $A_{11\cdot 2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $A_{22\cdot 1} = A_{22} - A_{21}A_{11}^{-1}A_{12}$

$$A^{-1} = \begin{bmatrix} A_{11\cdot 2}^{-1} & -A_{11\cdot 2}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}A_{11\cdot 2}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}A_{11\cdot 2}^{-1}A_{12}A_{22}^{-1} \end{bmatrix} \quad (*)$$

$$= \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}A_{22\cdot 1}^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22\cdot 1}^{-1} \\ -A_{22\cdot 1}^{-1}A_{21}A_{11}^{-1} & A_{22\cdot 1}^{-1} \end{bmatrix}$$

(proof)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22\cdot 1} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}}_{U}$$

* LDU decomposition of matrix A

$$\therefore A^{-1} = (LDU)^{-1} = U^{-1}D^{-1}L^{-1} = \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}\begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22\cdot 1}^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}$$

How to find L^{-1}. suppose that $L^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

$L \cdot L^{-1} = I$

$$\therefore \begin{bmatrix} I_{11} & 0 \\ A_{21}A_{11}^{-1} & I_{22} \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_{11}A & I_{11}B \\ A_{21}A_{11}^{-1}A + I_{22}C & A_{21}A_{11}^{-1}B + I_{22}D \end{bmatrix}$$

$$= \begin{bmatrix} I_{11} & 0 \\ 0 & I_{22} \end{bmatrix}$$ Solve the simultaneous equation.

$A = I_{11}$, $B = 0$, $C = -A_{21}A_{11}^{-1}$, $D = I_{22}$

$$\therefore L^{-1} = \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}$$

* UDL decomposition of matrix A in the precedent derivation of the inverse matrix would give (*)

Theorem 5 (Sherman - Morrison Lemma)

Let $a, b \in \mathbb{R}^n$ be vectors and $C \in \mathbb{R}^{n\times n}$ be an invertible matrix.

$$(I_n + ab^T)^{-1} = I_n - \frac{ab^T}{1 + b^Ta}$$

$$(C + ab^T)^{-1} = C^{-1} - \frac{C^{-1}ab^TC^{-1}}{1 + b^TC^{-1}a}$$

(proof)

$$(I + ab^T)\cdot\left(I - \frac{ab^T}{1+b^Ta}\right) = I - \frac{ab^T}{1+b^Ta} + ab^T - \frac{ab^Tab^T}{1+b^Ta}$$

$$= I - \frac{ab^T}{1+b^Ta} + ab^T - b^Ta\frac{ab^T}{1+b^Ta}$$

$$= I + \left(1 - \frac{1}{1+b^Ta}\right)ab^T - \left(\frac{b^Ta}{1+b^Ta}\right)ab^T$$

$$= I + \left(\frac{b^Ta}{1+b^Ta}\right)ab^T - \left(\frac{b^Ta}{1+b^Ta}\right)ab^T = I$$

$(C + ab^T)^{-1} = (I + a'b^T)^{-1}C^{-1}$, $a' = C^{-1}a$

$$= \left(I - \frac{a'b^T}{1+b^Ta'}\right)C^{-1} = \left(C^{-1} - \frac{C^{-1}ab^TC^{-1}}{1+b^TC^{-1}a}\right)$$